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Applications of the quantization in Physics and Noncommutative algebras

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1. General description of work

Relevance of the topic The work of this thesis is based on several almost independent problems on quantum generalized Heisenberg algebras 1.1, lattice W-algebras 1.2, Bergman's centralizer theorem 1.3 and on the algebraic torus actions 1.4, which can be summarized as follows:

1.1. **Quantum generalized Heisenberg algebras.**

A classical problem in quantum mechanics is the harmonic oscillator problem, whose solution relies on the representation theory of the Weyl algebra $\mathsf{A}_1(\mathbb{C})$ (defined below), which can be seen as the quotient of the enveloping algebra of the three-dimensional Heisenberg Lie algebra (defined below) by a suitable central element. Quantum analogues of the oscillator problem, of the Weyl algebra and of the enveloping algebra of the Heisenberg Lie algebra have been profusely studied (see e.g. [31]) in the last 30 years and several notions of quantum and deformed Heisenberg algebras have thence been proposed and studied.

Generalized Heisenberg Algebras (GHA, for short) were introduced in the physics literature in two different (but maybe with the same source) routes. One has been raised under the view of the Russian scientists perspective of the physics and mathematics and the other one from the Brazilian scientists point of view.

All has started from an algebra which is called the q-algebra or the algebra of q-deformed commutators (or the q-Oscillator algebra), which has been introduced by Ludvig Faddeev, Kulish and Jimbo independently in a series of articles related to the integrable models on quantum field theory and quantum spectral transform methods, between 1982-1989. The q-algebra is generated by operators a, $a^\dagger$ and N subject to the relations $[a, a^\dagger]_q = q^{-N}$, $[N, a] = -a$ and $[N, a^\dagger] = a^\dagger$, where $[a, a^\dagger] = aa^\dagger - qa^\dagger a$ and $\dagger$ stands for the Hermitian conjugate (adjoint or transpose; we work on matrix operator space) and N will be considered as a self adjoint matrix operator (the diagonal matrix with $n_{ii} = i$ for $i = 0, 1, \dots$) for q a fixed complex number. This algebra is related to quantum groups, whose properties have been intensively studied. They are applied in field theory and quantum gauge field theory.

In 1997, Sergey Yurievich Vernov and Melita Nikolaevna Mnatsakanova, have considered algebras of a more general form than the q-algebra, namely the algebras in which condition $[a, a^\dagger]_q = q^{-N}$ is replaced by $a^\dagger a = \varphi(N)$, $aa^\dagger = \varphi_1(N)$, where $\varphi(N)$ and $\varphi_1(N)$ are functions. They also have investigated the representations of these algebras in the case where φ and φ_1 are nonsingular. [42] (But it is possible to show that actually this algebra does not have any new thing to say by letting $q^{-N} = L$ and it is just an example of GWAs. But the most interesting result which people have gotten out of this example is that by solving these equations we will encounter with a new system of relations which will give us the q-analogues of the universal enveloping algebra of the 3 dimensional Heisenberg algebra $\mathfrak{h}_1$ which is the quotient of the free algebra $\mathbb{F}\left\langle a, a^\dagger, L\right\rangle$ modulo the two sided ideal I generated by the elements $aa^\dagger - qa^\dagger a - L$, $La^\dagger - q^{-1}a^\dagger L$ and $La - qaL$)

Then in 2000 and 2001, in a series of articles, Sergey and Melita, have considered a class of algebras which they called the generalized Heisenberg algebras. These algebras are almost the same as the generalized q-algebras studied in [46], with the only difference that in the generalized Heisenberg algebras (generalized q-algebras) (introduced by Vernov), they specified and fixed φ_1 in terms of φ i.e. they defined $\varphi_1(N) := \varphi(N+1)$. In this case, with a little care, we can see that, these algebras can be seen as the first Weyl algebra (example (1.3)) if we set $\varphi(N) = N - 1$. So this was the first appearance of the generalized Heisenberg algebras similar to their current format introduced in [16].

From the Brazilian side, in 1991, the first introduced generalization of the Heisenberg type (Heisenberg-Weyl type) algebras (algebras of the Canonical Commutation Relations (CCR)), has been introduced in a preprint in physics literature "Logistic algebras" (Which has not published officially, but it exists in the Internet in the archive of the Cern documents), where Rego-Monteiro, has tried to generalize the known algebraic structures appeared in the solution of the physical systems. He called them Logistic algebras, because of the logistic map used in his definition (In dynamical systems, the non invertible map $f(x) = rx(1-x)$, for $r \in \mathbb{R}$; is commonly called the "Logistic map". Now if we consider an another dynamical system $x_{n+1} = g(x_n)$ for an n-dimensional vector space x_n, which is an invertible map if we look at it as a differential operator and if we compose these two maps, then we will get the logistic map $x_{n+1} = rx_n(1-x_n)$, which has been considered by Rego-Monteiro for to define his algebras. And it is good to know that this map has been pointed out by Robert May (1976), as a simple idealized ecological model in the yearly variations in the population of an insect species).

In 2001, an alternative strategy to construct a solvable model working with selfadjoint Hamiltonians has been proposed by Evaldo Curado and Rego-Monteiro in a series of articles (as a generalization of logistic algebras) on what has been called generalized Heisenberg algebras (GHA), see [16] and the references therein. This strategy is mainly based on the existence of suitable intertwining and commutation relations, and on the existence of a certain function related to them. For instance we can see in q-algebra and in the deformed Heisenberg algebra introduced in [46][42], that the deformations of the canonical (anti-)commutation relations have proved to be quite useful for deducing eigenvalues and eigenvectors of certain Hamiltonians appearing in the literature devoted to non-Hermitian quantum mechanics. And as an application of the GHA's in the quantum field theory (QFT) is the idea of making the new generalized QFT (a new generalized solvable quantum model) by using these algebras, by replacing the characteristic function $f(h)$ in the relations (1.1) with $f(h) = th^2 + qh + s$. (We note that for $t = 0$ we will get the linear case $f(h) = qh + s$, which will give us the q-deformed Weyl algebra, discussed in the example (1.7))(see [16])

From the algebraic point of view, GHA have been studied mostly in [39], [38] and [35]. We start by recalling their definition and then enumerate some of the results which have been obtained thus far.

Let $f \in \mathbb{C}[h]$ be a fixed polynomial over $\mathbb{C}$. The *generalized Heisenberg algebra* $\mathcal{H}(f)$ is the unital associative $\mathbb{C}$-algebra with generators x, y, h satisfying the relations:

$$hx = xf(h), \quad yh = f(h)y, \quad yx - xy = f(h) - h. \tag{1.1}$$

In [39], the authors obtained the following results:

(i) Computation of the center $\mathsf{Z}(\mathcal{H}(f))$ of $\mathcal{H}(f)$ and of a basis for $\mathcal{H}(f)$ ([39, Theorem 4, Lemma 1]).
(ii) Isomorphism problem: necessary and sufficient conditions for $\mathcal{H}(f)$ and $\mathcal{H}(g)$ to be isomorphic ([39, Theorem 5]).
(iii) Determination of all finite-dimensional simple $\mathcal{H}(f)$-modules ([39, Theorem 12]).

Later, in [35], the following results were established:

(i) Necessary and sufficient conditions for $\mathcal{H}(f)$ to be Noetherian ([35, Proposition 2.1]).
(ii) Necessary and sufficient conditions for $\mathcal{H}(f)$ to be isomorphic to a generalized down-up algebra (see [13]) ([35, Corollary 2.1]).
(iii) Determination of all locally nilpotent and all locally finite derivations of $\mathcal{H}(f)$, in case $\mathsf{deg}\,(f) > 1$ ([35, Theorem 4.1, Corollary 4.1]).

(iv) Computation of the automorphism group of $\mathcal{H}(f)$, in case $\mathsf{deg}\,(f) > 1$ ([35, Theorem 5.1]).

Above, the emphasis on the case $\mathsf{deg}\,(f) > 1$ is justified by the fact that otherwise $\mathcal{H}(f)$ is isomorphic to a generalized down-up algebra and the latter algebras have been intensively studied since their initial introduction in [5] and generalization in [13]. In fact, this is one of the motivations for our proposed study: it has been revealed through a plenitude of research papers that the (generalized) down-up algebras form a very interesting class of algebras where, on the one hand, many properties can be tested and intuition can be developed more easily at the concrete level which they provide, and on the other hand, this class of algebras contains fundamental objects of algebraic interest, both at the classical and the quantum level. Striking examples are the enveloping algebra of the 3-dimensional complex simple Lie algebra $\mathfrak{sl}_2$ and of the 3-dimensional Heisenberg Lie algebra.

The class of GHA, although intersecting the class of generalized down-up algebras, contains also a wide range of algebras not in the latter class (this occurs precisely when $\mathsf{deg}\,(f) > 1$) and by [35], when a GHA is not a generalized down-up algebra, then it is not even a Noetherian ring, although it is a domain. So GHA include both the more tractable features of generalized down-up algebras and the less chartered features of non-Noetherian domains.

In summary, the Weyl algebra, which has been discussed in example (1.3), is the special case of the defining relations given by (1.1), where instead of taking the relations given by $f(h)$, we consider the one for $f(h) = h - 1$.

Another well-developed and well-studied class of algebras related to GHA is that of Generalized Weyl Algebras (GWA, for short), introduced by Bavula over a series of papers (see e.g. [3]). Let A be an algebra (over the field $\mathbb{F}$), σ an algebra automorphism of A and $z \in \mathsf{Z}(A)$ a central element of A. Then the GWA $B := A(\sigma, z)$ is the unital associative algebra generated by A and elements x, y, subject to the relations:

$$xa = \sigma(a)x, \quad ay = y\sigma(a), \quad yx = z, \quad xy = \sigma(z), \qquad \forall a \in A. \tag{1.2}$$

Examples of GWA are the Weyl algebra $A_1(\mathbb{F})$, the quantum plane $\mathbb{F}_q[x, y]$, the quantum Weyl algebra $A_1^q(\mathbb{F})$, the enveloping algebra of the 3-dimensional Heisenberg Lie algebra and its quantum analogue (all defined next). Other examples which we will not define but are nevertheless very important are the Noetherian (generalized) down-up algebras mentioned above (see [13]), the enveloping algebra of the 3-dimensional complex simple Lie algebra $\mathfrak{sl}_2$, its primitive infinite-dimensional quotients and the quantum analogues of these, and the Smith algebras (see e.g. [4] for more details).

Example 1.3 (The first Weyl algebra)**.** The Weyl algebra $A_1(\mathbb{F})$ is the unital associative algebra generated over $\mathbb{F}$ by elements u, v, subject only to the relation $vu - uv = 1$. This algebra, along with its higher degree analogues $A_n(\mathbb{F}) := A_1(\mathbb{F})^{\otimes n}$, are of paramount importance in representation theory as they control the representations of the finite-dimensional complex nilpotent Lie algebras.

Let $A = \mathbb{F}[h]$ be a polynomial algebra in the variable h. Consider its automorphism σ defined by $\sigma(h) = h - 1$ and the element $z = h$. Then the GWA $A(\sigma, z)$ is isomorphic to $A_1(\mathbb{F})$. Indeed, let $\mathbb{F}\langle U, V\rangle$ be the free algebra on the generators U, V. By its universal property, there is a uniquely defined (unital) algebra homomorphism

$$\mathbb{F}\langle U, V\rangle \longrightarrow \mathbb{F}[h](\sigma, h), \qquad U \to x,\ V \to y. \tag{1.4}$$

The image of $VU - UV - 1$ under this map is $yx - xy - 1 = h - \sigma(h) - 1 = 0$, so the above map induces an algebra homomorphism defined on the quotient of $\mathbb{F}\langle U, V\rangle$ by its two-sided ideal $(VU - UV - 1)$, which is precisely $A_1(\mathbb{F})$, by definition. We call this new map $\phi : A_1(\mathbb{F}) \longrightarrow \mathbb{F}[h](\sigma, h)$ and denote the cosets of U and V in the factor algebra $A_1(\mathbb{F}) = \mathbb{F}\langle U, V\rangle/(VU - UV - 1)$ by u and v, respectively. This map ϕ is clearly surjective, as the image contains the generators $x = \phi(u)$, $y = \phi(v)$ and $h = \phi(vu)$. At this point, further knowledge on the structures of $A_1(\mathbb{F})$ and $\mathbb{F}[h](\sigma, h)$ (namely, suitable bases) would be needed to conclude that ϕ is injective. This can be avoided by defining a new algebra homomorphism $\psi : \mathbb{F}[h](\sigma, h) \longrightarrow A_1(\mathbb{F})$ with the property that $\psi \circ \phi$ acts as the identity on the generators u, v of $A_1(\mathbb{F})$. This can be done in a similar fashion: define a map

$$\mathbb{F}\langle X, Y, H\rangle \longrightarrow A_1(\mathbb{F}), \qquad X \to u,\ Y \to v,\ H \to vu \tag{1.5}$$

and show that the elements corresponding to the defining relations (1.2) of $\mathbb{F}[h](\sigma, h)$ are in the kernel. For example, for the relation $xa = \sigma(a)x$ for all $a \in \mathbb{F}[h]$, it is enough to take $a = h$ and consider the image of the element $XH - (H - 1)X$, which is $u(vu) - (vu - 1)u = u(vu) - (uv)u = 0$, as desired. Similarly we conclude that this map factors through the desired map ψ, proving the isomorphism.

Example 1.6 (Quantum plane)**.** The quantum plane (Also known as the Manin's quantum plane) is the quotient of the free algebra $\mathbb{F}\langle x, y\rangle$ modulo the two sided ideal generated by the element $xy - qyx$, for $q \neq 0, 1$, root of unity.

It is a GWA in the following way. Suppose we are given $\mathbb{F}_q[x, y]$. We want to find the GWA $A(\sigma, z)$ subject to the relations $xy = \sigma(z)$, $yx = z$, $xa = \sigma(a)x$, and $ya = y\sigma(a)$ for $z \in Z(A)$ and for all $a \in A$, isomorphic to $\mathbb{F}_q[x, y]$.

For to do this, we need to find an automorphism σ and $z \in Z(A)$ and a suitable base ring A, such that the condition $xy - qyx = 0$ satisfies.

We note that the only thing we know is $xy - qyx = 0$. By using the relation $yx = z$, we get $qyx = qz$. Now by subtracting both sides of the $qyx = qz$ of $xy = \sigma(z)$, we get $0 = xy - qyx = \sigma(z) - qz$, which means that $\sigma(z) = qz$ for any $z \in Z(A)$ and this equation can be the only condition which we can have for σ. We can see this by comparing the relation $xy = qyx$ in $\mathbb{F}_q[x, y]$ with the relations $xa = \sigma(a)x$ and $ay = y\sigma(a)$ of the proposed GWA $A(\sigma, z)$ for any $a \in A$. Now if for $a \in A$ we have $a = z \in Z(A)$, then we get that $xz = qzx$ and $yz = qzy$, which are redundant and we can get them out of the relations $xy = qz$ and $yx = z$ of the GWA $A(qz, z)$. Now if $a \neq z$ or not equal to any combination of z, then the relation which we have found for σ will not be the only relation which we need to define σ, which is a contradiction with our early result that it is enough for to define our automorphism. So by choosing $z = a$ for any $a \in A$, the only choice for A becomes $A = \mathbb{F}[z]$ and the correspondence $x \leftrightarrow x$, $y \leftrightarrow y$ and $yx \leftrightarrow z$, will give us the desired isomorphism, in a similar way as in the example (1.3).

Here we also could use the similar way as in the example 1.7. But I just wanted to find it directly for to see how can it be done for a simple example. One can follow (1.7).

Example 1.7 (Quantum Weyl algebra)**.** The quantum Weyl algebra $\mathcal{A}_1^q(\mathbb{F})$ where $q \in \mathbb{F}\setminus\{0\}$ is the algebra with generators x, y and defining relation $xy - qyx = 1$.

It is a GWA in the following way. Let $A = \mathbb{F}[t]$, let $\sigma : A \to A$ be the automorphism defined by $\sigma(f(t)) = f(qt + 1)$ for any $f(t) \in A$, and let z be the polynomial $p(t) = t \in A$. Let $B = A(\sigma, z)$ be the corresponding GWA. Then relations $(*)$ in the example 1.6 imply that $yx = t$ and $xy = qt + 1$ so that $xy - qyx = 1$ holds.

From $xt = x(yx) = (xy)x = (qt + 1)x$ follows that $xt^k = (qt + 1)^k x$ for any $k \in \mathbb{Z}_{\geq 0}$ and thus, by linearity, $xf(t) = f(qt + 1)x$ for any polynomial $f(t) \in A$. Analogously $f(t)y = yf(qt + 1)$ for any $f(t) \in A$. This shows that relations $xa = \sigma(a)x$ and $ay = y\sigma(a)$

are redundant and that B generated by x, y with the single relation $xy - qyx = 1$. Thus B is isomorphic to the quantum Weyl algebra $\mathcal{A}_1^q(\mathbb{F})$.

If we take $q = 1$, we get the example (1.3).

Example 1.8 (enveloping algebra of the 3-dimensional Heisenberg Lie algebra)**.** The first Heisenberg algebra $\mathfrak{h}_1$ is an associative Lie algebra of dimension 3 that is algebraically generated by the generators X, Y and H which are subject to the Lie bracket relations $[X, Y] = H$ and $[X, H] = 0 = [Y, H]$. Hence H is a central element. And its universal enveloping algebra $U(\mathfrak{h}_1)$ is the quotient of the free algebra $\mathbb{F}\langle X, Y, H\rangle$ modulo the two sided ideal I generated by elements $XY - YX - H$, $XH - HX$ and $YH - HY$.

It is a GWA in the following way. Let $A = \mathbb{F}[h, z]$ be the commutative polynomial ring in two variables, and let $\sigma : A \to A$ be the automorphism defined by $\sigma(z) = z - h$ and $\sigma(h) = h$. Now let $A(\sigma, z)$ be the corresponding GWA. The relations $yx = z$ and $xy = \sigma(z)$ of (1.2) and our definition of σ imply that $yx = z$ and $xy = z - h$. So $yx - xy = h$.

Also relations $xa = \sigma(a)x$ and $ay = y\sigma(a)$ of (1.2) for every $a \in A = \mathbb{F}[h, z]$ will imply that $xh = \sigma(h)x = hx$, $hy = y\sigma(h) = yh$ and $xz = \sigma(z)x = zx - hx$, $yz = \sigma(z)y = zy - hy$.

We claim that the later relations coming from z are redundant, and we can get them from the relation $yx - xy = h$. For to see this, let us multiply both sides of $yx - xy = h$ with x from the right side. So we have $yxx - xyx = hx = zx - xz$ and in a similar way we can get $hy = zy - yz$. So the correspondence $X \leftrightarrow x$, $Y \leftrightarrow y$, and $H \leftrightarrow h$, will give us the isomorphism $U(\mathfrak{h}_1) \cong \mathbb{F}[h, z](\sigma, z)$, in a similar way as in the example (1.3).

Example 1.9 (quantum Heisenberg algebra)**.** The quantum Heisenberg algebra $\mathcal{H}_q$, is the free algebra generated by X, Y and H subject to the relations $XH = q^2HX$, $YH = q^{-2}HY$ and $XY - q^{-2}YX = q^{-1}H$, for $q \in \mathbb{F}$ s.t. $q^4 \neq 1$. Hence $\mathcal{H}_q$ will be the quotint of the free algebra $\mathbb{F}\langle X, Y, H\rangle$ modulo the two sided ideal I generated by elements $XH - q^2HX$, $YH - q^{-2}HY$ and $XY - q^{-2}YX - q^{-1}H$.

It is a GWA in the following way. Take $\theta = XY - q^2YX$. Then by some calculation we can see that $X\theta = q^{-2}\theta X$, $Y\theta = q^2\theta Y$ and $H\theta = \theta H$. Now let us set $C = H\theta$. Then we see that $HC = CH$, $XC = CX$ and $YC = CY$ by knowing that $XH = q^2HX$, and $YH = q^{-2}HY$. This means that $C \in Z(\mathcal{H}_q)$. Now by knowing this, let us set $z = \theta - q^{-1}H \in Z(\mathbb{F}[H, \theta])$ and let us define automorphism $\sigma : F[H, \theta] \to F[H, \theta]$ such that it sends $H \mapsto q^2H$ and $\theta \mapsto q^{-2}\theta$ by getting used of the relations $XH = q^2HX$ and $YH = q^{-2}HY$. And then it become easy to check that $\mathcal{H}_q \simeq \mathbb{F}[H, \theta](\sigma, z)$ by correspondence $X \leftrightarrow X$, $Y \leftrightarrow Y$ and $\theta - q^{-1}H \leftrightarrow YX$ in a similar way as in the example (1.3).

There are many other important examples of GWA's. But we cannot include them here due to the limitations in the number of pages.

The above examples show that GWA include a variety of important examples in mathematics and theoretical physics. Let us look at a particular motivating example of interest.

Example 1.10 (towards weak GWA)**.** Let $A = \mathbb{F}[h, \omega]$ be a polynomial ring in two (commuting) variables and fix a polynomial $f(h) \in \mathbb{F}(h)$ of degree 1. Take $z = h + \omega$ and σ to be the automorphism of A determined by $\sigma(h) = f(h)$ and $\sigma(\omega) = \omega$ (σ is bijective precisely by the assumption that $\deg f = 1$). Then the GWA $A(\sigma, z)$ is the $\mathbb{F}$-algebra generated by elements h, ω, x, y, subject to the relations (1.2), which can be written as follows, after a slight simplification:

$$xh = f(h)x, \quad hy = yf(h), \quad yx = h + \omega, \quad xy = f(h) + \omega, \quad [x, \omega] = 0 = [y, \omega], \tag{1.11}$$

where $[a, b] = ab - ba$ is the commutator. From (1.11), we notice that the central generator ω is unnecessary as we deduce that $xy - f(h) = \omega = yx - h$. Moreover, the resulting relation

$xy - f(h) = yx - h$ is consistent with the relations $[x, \omega] = 0 = [y, \omega]$ and thus we deduce that the GWA $A(\sigma, z)$ is the $\mathbb{F}$-algebra generated by elements h, x, y, subject to the relations:

$$xh = f(h)x, \quad hy = yf(h), \quad xy - yx = f(h) - h. \tag{1.12}$$

Comparing the above with the defining relations (1.1) for the GHA $\mathcal{H}(f)$, we see that these are the same, except that the roles of x and y are switched. This shows that $\mathcal{H}(f)$ is a GWA **in case** $\deg f = 1$.

In the above example, the restriction that $\deg f = 1$ was necessary only to guarantee that σ is bijective. If one abandons such a restriction in the definition of a GWA, then we obtain what has been called in [38] a *weak generalized Weyl algebra* (wGWA, for short). Concretely, given an algebra A, $z \in \mathsf{Z}(A)$ and an *endomorphism* σ of A, the wGWA $B := A(\sigma, z)$ is the unital associative algebra generated by A and elements x, y, subject to the GWA relations (1.2) above. In particular, taking an arbitrary polynomial $f(h) \in \mathbb{F}[h]$ in the example above shows that all GHA are wGWA. However, the weaker condition that σ is not necessarily bijective carries nontrivial implications: for a Noetherian domain A and $0 \neq z \in \mathsf{Z}(A)$, the wGWA $A(\sigma, z)$ may no longer be a domain or even Noetherian, as is shown in [35, Proposition 2.1]. Therefore, the further study of GHA will also produce a better understanding of wGWA in general.

As of the present, two of the most successful achievements in the classification of all simple representations of noncommutative infinite-dimensional associative algebras are still the works of Block [12] and Bavula [2], where the simple modules for the Weyl algebra $A_1(\mathbb{F})$, the enveloping algebras of $\mathfrak{sl}_2$ and of the 3-dimensional Heisenberg Lie algebra, the quantum plane and the quantum Weyl algebra are classified, in a certain sense. The latter are all examples of GWA, as argued before. The classification of the simple modules for GHA would be a very nice extension of the theory beyond the scope of GWA.

Because of the difficulty of classifying all simple representations, especially of infinite-dimensional associative algebras, many studies have focused on specific types of representations. Examples (we shall not define the terminology here) are finite-dimensional modules, highest weight modules, weight modules with finite-dimensional weight spaces, Whittaker modules, Gelfand-Zetlin modules, and more. Recently, some authors have also studied torsion free actions on a certain subalgebra (typically a commutative polynomial subalgebra; in the case of semisimple Lie algebras, the enveloping algebra of the Cartan subalgebra is usually chosen) and the results obtained are interesting and seem appropriate for generalization (see e.g. [40]).

Another important direction in representation theory is the classification of primitive ideals (i.e., annihilators of simple representations). This idea has successfully been put forward by Dixmier, in light of the fact that in general it is considered a wild problem to classify the simple representations themselves. In contrast, the classification of primitive ideals of enveloping algebras of Lie algebras has motivated a lot of interesting connections in mathematics, two of the best such examples being the Dixmier map and the so-called Dixmier-Moeglin equivalence. Both of these techniques have proven to be extremely useful also in the study of primitive ideals of quantum algebras (see [28].

1.1.1. ***Goals and objectives of the work toward qGHAs.*** Our work aims to generalize this to a larger class of algebras which we call quantum generalized Heisenberg algebras. These depend on an arbitrary base field $\mathbb{F}$, a *quantum* parameter $q \in \mathbb{F}^*$ and two polynomials

f and g. Our main motivation for introducing a generalization of this class, besides providing a broader framework for the investigation of the underlying physical systems, comes from the observation in [35] that the classes of generalized Heisenberg algebras and of (generalized) down-up algebras intersect (see the seminal paper [5] on down-up algebras and also [13]), although neither one contains the other. In spite of the name, the class of generalized Heisenberg algebras does not include the enveloping algebra of the Heisenberg Lie algebra nor its quantum deformations introduced in [31], nor the enveloping algebra of $\mathfrak{sl}_2$. These and many other like algebras are now included in the class of *qGHA*s. The other interesting feature of our study comes from the fact that quantum generalized Heisenberg algebras are generically non-Noetherian and we believe that there are yet not enough studies into the representation theory of non-noetherian algebras which are somehow related to deformations of enveloping algebras of Lie algebras, as is the case with quantum generalized Heisenberg algebras.

Definition 1.13. Let $\mathbb{F}$ be an arbitrary field. Then for any fixed $f, g \in \mathbb{F}[h]$ and $q \in \mathbb{F}^*$, the quantum generalized Heisenberg algebra (*qGHA*, for short) $\mathcal{H}_q(f,g)$ is the unital associative algebra over $\mathbb{F}$ generated by x, y, and h subject to the defining relations

$$hx = xf(h), \quad yh = f(h)y, \quad yx - qxy = g(h). \tag{1.14}$$

Consider the 3-dimensional Lie algebra $\mathfrak{sl}_2$, with basis elements x, y, h and Lie bracket given by $[x,h] = 2x$, $[h,y] = 2y$ and $[y,x] = h$. We can view its enveloping algebra as *qGHA* $\mathcal{H}_1(h-2,h)$. In the representation theory of $\mathfrak{sl}_2$, x and y are often represented as raising and lowering operators on a finite or countable vector space. For example, in [**?**] and [**?**] the existence of such operators on the vector space whose distinguished basis is a suitably defined poset is used to solve important combinatorial problems.

Take $V = \mathbb{F}[t^{\pm 1}]$, the Laurent polynomial algebra, and suppose that x and y act on V as raising and lowering operators, respectively, so that h acts diagonally. We can assume that, relative to the basis $\left\{t^k\right\}_{k\in\mathbb{Z}}$, we have

$$xt^k = t^{k+1}, \quad yt^k = \mu(k)t^{k-1} \quad \text{and} \quad ht^k = \lambda(k)t^k, \qquad \text{for all } k \in \mathbb{Z}, \tag{1.15}$$

where $\lambda, \mu : \mathbb{Z} \longrightarrow \mathbb{F}$. Then, the $\mathfrak{sl}_2$ relations impose the conditions

$$\mu(k+1) = \mu(k) + \lambda(k) \quad \text{and} \quad \lambda(k+1) = \lambda(k) - 2,$$

so $\lambda(k+1)$ is affine on $\lambda(k)$ and $\mu(k+1)$ is linear on $\mu(k)$ and $\lambda(k)$.

The 3-dimensional Heisenberg Lie algebra has basis x, y, h and Lie brackets $[h,x] = [h,y] = 0$ and $[y,x] = h$. Its enveloping algebra can be seen as the *qGHA* $\mathcal{H}_1(h,h)$. Then, the Heisenberg relations imposed on (1.15) give

$$\mu(k+1) = \mu(k) + \lambda(k) \quad \text{and} \quad \lambda(k+1) = \lambda(k),$$

so λ is constant and $\mu(k+1)$ is affine on $\mu(k)$.

Another related example is given by the algebras similar to the enveloping algebra of $\mathfrak{sl}_2$ introduced by Smith in [**?**]. These are precisely the *qGHA* of the form $\mathcal{H}_1(h-1,g)$, for $g \in \mathbb{F}[h]$. The corresponding conditions imposed on (1.15) by the Smith algebra relations are

$$\mu(k+1) = \mu(k) + g(\lambda(k)) \quad \text{and} \quad \lambda(k+1) = \lambda(k) - 1,$$

so $\lambda(k+1)$ is affine on $\lambda(k)$ but now $\mu(k+1) - \mu(k)$ is polynomial on $\lambda(k)$.

As a final example, if we take a generalized Heisenberg algebra, i.e. a *qGHA* of the form $\mathcal{H}_1(f, f-h)$, then the corresponding conditions imposed on (1.15) are

$$\mu(k+1) - \mu(k) = \lambda(k+1) - \lambda(k) \quad \text{and} \quad \lambda(k+1) = f(\lambda(k)),$$

so λ and μ differ by a constant and $\lambda(k+1)$ is polynomial in $\lambda(k)$.

With the more general relations allowed for by our definition of a *qGHA*, we can include all of the above cases and, more generally, in $\mathcal{H}_q(f,g)$ we have

$$\mu(k+1) = q\mu(k) + g(\lambda(k)) \quad \text{and} \quad \lambda(k+1) = f(\lambda(k)),$$

so that $\lambda(k+1)$ is polynomial in $\lambda(k)$ and $\mu(k+1)$ is affine in $\mu(k)$ and polynomial in $\lambda(k)$. Representations of the *qGHA* $\mathcal{H}_q(f,g)$ will thus classify the creation and annihilation operators as in (1.15), under the latter assumptions.

In the papers [36] and [37], we have found some results concerning the classification of all simple finite dimensional $\mathcal{H}_q(f,g)$-modules, determining automorphism groups of the $\mathcal{H}_q(f,g)$ when $\deg f > 1$ and we also solved the isomorphism problem for this class of algebras and we have determined when a quantum generalized Heisenberg algebra is Noetherian and many other ring-theoretical properties like Gelfand-Kirillov dimension and being domain. But the class of quantum generalized Heisenberg algebras seems very interesting and worth studying further, such as computing their global dimensions (compare [13]) and determining those quantum generalized Heisenberg algebras all of whose finite-dimensional representations are completely reducible (compare [**?**]). Studying simple weight modules for $\mathcal{H}_q(f,g)$ and determining the primitive ideals of $\mathcal{H}_q(f,g)$ (compare [**?**] and [**?**]). Investigating Hochschild (co)homology of $\mathcal{H}_q(f,g)$ (compare [15]). Investigating when $\mathcal{H}_q(f,g)$ has a Hopf algebra structure can be very interesting as in [**?**] the authors embed a certain type of down-up algebra A into a skew group algebra $A * G$, where G is a subgroup of the automorphism group of A, and then construct a Hopf algebra structure on $A * G$. As an example, when the defining parameters of A are such that A is isomorphic to the enveloping algebra of the Heisenberg Lie algebra, then the group G is trivial and the Hopf structure obtained agrees with the usual one on an enveloping algebra. And in a very different direction, in [**?**] the authors classify the down-up algebras which have a Hopf algebra structure and we also have done the same for some subclasses of the class of quantum generalized Heisenberg algebras and what remains is to compute the tensor product modules of the simple modules of these algebras.

One can also pursue a more geometric standpoint, by looking at Poisson algebras which can be associated with generalized Heisenberg algebras via the semiclassical limit process (see [28]). The study of these Poisson algebras could be relevant to the original setting where generalized Heisenberg algebras were defined, motivated by questions in mathematical physics. Moreover, it would be interesting to study possible correspondences between the properties of the latter Poisson algebras and the properties of *qGHA*, especially concerning representations, primitive ideals, symplectic cores (see [28]) as well as Hochschild and Poisson cohomology. In light of [41], it is expected that these should be related.

On the other hand, going back to the Physics literature on generalized Heisenberg algebras, where these appeared to be defined over rings more general than the polynomial ring $\mathbb{F}[h]$, it would also be of interest to consider quantum generalized Heisenberg algebras defined over a Laurent polynomial ring $\mathbb{F}[h^{\pm 1}]$, a power series ring $\mathbb{F}[[h]]$ or the rational function field $\mathbb{F}(h)$.

1.1.2. ***Goals and objectives of the work toward GHAs.*** The plan is to initiate a thorough study of GHA from several points of view which we will summarize below. Whenever feasible, we will try to formulate the results in the broader context of wGWA, compare them with known results on GWA and on generalized down-up algebras.

To facilitate clarity, we will subdivide our exposition into our areas of interest. It should be understood that we are just now delving into this subject, so these serve as guidelines only for our research and the concrete accomplishments in our thesis will naturally depend on our progress along each of these subjects and on the positive results that we get along the way.

1.1.3. *Representation theory.* Representation theory is of great importance in almost all areas of pure and applied mathematics, and in theoretical physics. In spite of this, it is generally considered a hopeless problem to classify all simple representations for a given infinite-dimensional algebra, except under suitably special conditions.

As has been indicated above, all finite-dimensional simple $\mathcal{H}(f)$-modules have been classified in [39] and in [38] the authors consider the problem of classifying the simple weight modules over weak generalized Weyl algebras over a polynomial ring in just one variable. Although the latter does not directly cover GHA, partial results about infinite-dimensional simple weight modules for GHA have been obtained in [38].

Given the already mentioned prominence of representation theory, and the specific open problem raised in [39] (and still left open in [38]) of determining other, if not all, classes of simple representations of $\mathcal{H}(f)$, our first and foremost topic of interest will be to address this open problem. We will mostly try two routes:

(i) Examine the arguments in [12] and [2] and see if we can adapt their methods to the case of GHA.
(ii) Study particular classes of representations of $\mathcal{H}(f)$, starting with an attempt to arrive at a good definition of Whittaker modules and of torsion free module structures on suitable commutative subalgebras of $\mathcal{H}(f)$. The latter will involve studying the methods employed in [40] and also the references to other types of representations included in this paper.

1.1.4. *GHA over fields of positive characteristic and extension to $f \in \mathbb{F}[[h]]$.* Many properties of an algebra tend to depend on the characteristic of the base field. Typically, but not exclusively, when the ground field has positive characteristic, the center of the algebra is usually bigger (in some sense) than the center of the same algebra over a field of characteristic 0. In fact, it is common for an algebra over a field of prime characteristic to satisfy a polynomial identity, as is the case of the Weyl algebra $\mathsf{A}_1(\mathbb{F})$, which is simple with trivial center when $\mathsf{char}(\mathbb{F}) = 0$ but becomes a free module of finite rank over its center (a polynomial algebra in two variables) when $\mathsf{char}(\mathbb{F}) = p > 0$. This has strong implications on the structure and representation theory of the Weyl algebra.

In the specific example of GHA, these have always been considered over the field $\mathbb{C}$ of complex numbers. It is likely that most known results about these algebras still hold over a (possibly algebraically closed) field of characteristic 0, but we expect to obtain different results in case the base field has positive characteristic. The analogous question for down-up algebras has been taken up in [30].

Another direction would be to go back to the original definition from the physics literature [16] where $f(h)$ was taken to be an analytic function. Algebraically, this corresponds to taking $f \in \mathbb{F}[[h]]$, where $\mathbb{F}[[h]]$ is the ring of formal power series in h. This approach could lead to interesting variations of the theory, as the group of units in $\mathbb{F}[[h]]$ is much larger than that of $\mathbb{F}[h]$.

1.1.5. *Endomorphisms of GHA.* One of the longest-standing open conjectures in algebra is the *Dixmier conjecture* which asks if every endomorphism of the Weyl algebra $A_1(\mathbb{F})$, where

$\mathbb{F}$ is a field of characteristic 0, or more generally of the n-th Weyl algebra $A_n(\mathbb{F}) = A_1(\mathbb{F})^{\otimes n}$ is an automorphism. This problem remains open even for $A_1(\mathbb{F})$ and in general Tsuchimoto and, independently, Belov-Kanel and Kontsevich, proved that the Dixmier conjecture is stably equivalent to the Jacobian conjecture from the field of algebraic geometry.

It is natural to ask similar questions regarding other algebras. Note that in characteristic 0 the Weyl algebras $A_n(\mathbb{F})$ are simple, so any endomorphism of the Weyl algebra is necessarily injective. Thus, a natural generalization of this question for other families of non-simple algebras is to ask whether all monomorphisms are automorphisms. This type of question has been answered for several classes of algebras, some of which are GWA and some not, but to our knowledge it hasn't been investigated for GHA. Given the relevance of the Dixmier conjecture, this could be an interesting problem to study within the class of GHA.

1.1.6. *The semiclassical limit and a Poisson algebra analogue of GHA.* In what follows we will continue to use the commutator notation $[a, b] = ab - ba$. A Poisson algebra P is a commutative (associative) algebra equipped with a bilinear product $\{\,,\}$ such that $(P, \{\,,\})$ is a Lie algebra and the map $\{a, -\} : P \times P \longrightarrow P$ is a derivation of P with respect to its associative product, for all $a \in P$.

We start out by briefly recalling the *semiclassical limit* process. Further details can be found in [28]. Let A be an algebra and $\mathcal{R} \subseteq \mathsf{Z}(A)$ be a central subalgebra. Assume further that $ra = 0$ for $r \in \mathcal{R}$ and $a \in A$ implies that either $r = 0$ or $a = 0$. Choose $0 \neq \hbar \in \mathcal{R}$ which is not a unit in A. Since $\hbar$ is a central non-unit, $\hbar A$ is a proper ideal of A. We will use any of the notations $\overline{a}$, $a + \hbar A$ or the more suggestive $a|_{\hbar=0}$ to denote the image of $a \in A$ under the canonical map onto $\overline{A} = A/\hbar A$.

Assume that $\overline{A}$ as above is commutative. Then we can define a Poisson bracket on $\overline{A}$ by setting

$$\{\overline{a}, \overline{b}\} = \overline{\hbar^{-1}[a, b]} = (\hbar^{-1}[a, b])\big|_{\hbar=0}, \quad \forall a, b \in A, \tag{1.16}$$

where $\hbar^{-1}[a, b]$ just denotes the unique element $\gamma(a, b) \in A$ such that $[a, b] = \hbar\gamma(a, b) \in \hbar A$ (the existence of such an element follows from the commutativity of $\overline{A}$ and the uniqueness from the fact that $\hbar \neq 0$ is not a zero divisor in A). Indeed, it is straightforward to check that (1.16) is independent of the choice of representatives $a, b \in A$ and defines an $\mathcal{R}$-bilinear Poisson bracket on $\overline{A}$. Endowed with this bracket, the Poisson algebra $\overline{A}$ is called the semiclassical limit of A and, in turn, A is called a quantization of $\overline{A}$.

Example 1.17. Consider the quantum plane over $\mathbb{F}[[\hbar]]$ as the $\mathbb{F}[[\hbar]]$-algebra A generated by x and y, subject to the relation $yx = qxy$, where $q = e^{\hbar} := \sum_{k\geq 0} \frac{\hbar^k}{k!} = 1 + \hbar \sum_{k\geq 1} \frac{\hbar^{k-1}}{k!}$. Then $[y, x] = (q - 1)xy \in \hbar A$, so $\overline{A} = A/\hbar A \cong \mathbb{F}[x, y]$ is the commutative polynomial ring in two variables (which we still denote by x and y) and it becomes a Poisson algebra under the semiclassical limit process described above. The Poisson bracket is given by $\{y, x\} = xy$, and for all $f, g \in \overline{A}$ we have $\{f, g\} = \sum\limits_{i,j=1}^{n} \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\{x, y\} = \sum\limits_{1\leq j<i\leq n} (\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x})\{x, y\}$.

We can construct a GHA replacing the field $\mathbb{F}$ with $\mathbb{F}[[\hbar]]$ and, under certain restrictions, obtain a Poisson algebra as a semiclassical limit. This can lead to an abstract definition of a Poisson GHA which relates to a GHA in a similar way that Poisson polynomial rings relate to Ore extensions and that the Poisson generalized Weyl algebra defined in [14] relates to the corresponding quantum generalized Weyl algebra (see [14] for details). The relevance of this is that often the Poisson semiclassical limit carries important information about its quantizations, as is illustrated in [28]. This is especially relevant at the level of representation theory and in the classification of primitive ideals, and we can do no better than to cite from [28, p. 167]:

> If A is a generic quantized coordinate ring of an affine algebraic variety V over an algebraically closed field of characteristic zero, and if V is given the Poisson structure arising from an appropriate semiclassical limit, then the spaces of primitive ideals in A and symplectic cores in V, with their respective Zariski topologies, are homeomorphic.

The quoted text above is a conjecture which has motivated a lot of recent research (see e.g. [27]) and we are interested in exploring this route for GHA.

1.1.7. *Hochschild (respectively Poisson) cohomology of GHA (and their Poisson algebra analogues).* The Hochschild cohomology of an associative algebra A is a useful and important invariant, although it is generally difficult to compute. In [15] the authors computed the Hochschild homology and cohomology of a certain class of down-up algebras. Given the already mentioned relation between GHA and down-up algebras, we consider it to be an interesting problem to tackle the Hochschild cohomology of GHA. In order to approach this problem, we need to investigate the methods used in [15] and also methods used elsewhere (by A. Solotar et al.) for the computation of Hochschild cohomology of certain classes of GWA over a polynomial ring in one variable.

Our interest in this problem stems from the following:

(a) The first Hochschild cohomology group is the Lie algebra of outer derivations and [35] offers a classification of the locally-finite derivations of a GHA. Thus, it would be a natural step to continue the work in [35] and determine this Lie algebra fully.
(b) The second Hochschild cohomology group controls the infinitesimal deformations of the underlying algebra, a problem which fits in naturally with our research.
(c) In general, the Hochschild cohomology of an algebra has a rich additional structure given by the cup product and the Gerstenhaber bracket, making it into a Gerstenhaber algebra (loosely speaking, this is a kind of graded Poisson structure relative to the cup product and the Gerstenhaber bracket). This structure is generally very difficult to compute in concrete examples, since it is defined in terms of the bar resolution of the algebra, which is not practical as a computational method. However, very recent methods have emerged to help compute this structure in terms of an arbitrary resolution. Testing these methods on the class of GHA could reveal interesting results towards both the understanding of GHA and the methods themselves.

In connection with the Poisson semiclassical limit of a GHA, as explained in 1.1.6, it could also be interesting to compare the results on Hochschild cohomology of GHA with the Poisson cohomology of the associated semiclassical limit, in the spirit of [41].

Author's works on the topic of qGHAs:

References

[1] Lopes, Samuel A., and Farrokh Razavinia. *Quantum generalized Heisenberg algebras and their representations.* Communications in Algebra (2021): 1-21.

[2] Lopes, Samuel A., and Farrokh Razavinia. *Structure and isomorphisms of quantum generalized Heisenberg algebras.* Journal of Algebra and its Applications (2021).

1.2. **Lattice W-algebras.** In 1985 first example of W_3 algebras has been introduced by Alexander Zamolodchikov in the investigasion for the possibility of existence of new additional infinite symmetries in the context of two-dimensional Conformal Field Theory [47], and as the possible extention of Virasoro algebra. Vladimir Fateev and Zamolodchikov [22] found the bosonic representation for W_3 algebras and noted some connection with $\mathfrak{sl}_3$ Lie algebra. In a series of articles [23, 24, 25], Fateev and Lukyanov have shown that there exist W-algebras associated to every simple Lie algebra and found the bosonic representation of generatores in W-algebras. They discovered that free bosonic representation of W-algebras is given by quantum Miura transformation, classical analoge of which was well-known in the theory of integrable non-linear evalution of Korteweg-de Vries type [20]. In the spirit of work [19], Virasoro algebra should commute (in the Feigin-Fuchs representation) with screening operators. As the matter of fact this property was given in the works [47, 25, 7] as the main mathematical background of such a definition of W-algebras was developed in the works [7, 26, 8] where it was shown that W-algebras are the result of quantum Drinfeld-Sokolov reduction of K-M. algebras. As in [11] have been shown that screening operators satisfy to the quantum Serre relation, i.e. they constitute the nilpotent part of quantum groups. So mathematically speaking we have

$$W \simeq InvU_q(\mathfrak{n}_+), \tag{1.18}$$

where $g = \mathfrak{n}_+ \oplus h \oplus \mathfrak{n}_-$ is the Lie algebra associated to W-algebra. In our work, we describe some variant of lattice analogus of W-algebras, given by definition 1.18. First example of classical lattice W_2 algebra (lattice Virasoro algebra) was found by Faddeev and Takhtajan in the work [21] under their studying of Liouville model on the lattice. Quantum analogue of Faddeev-Takhtajan algebra was obtained by Volkov in 1992. Boris L. Feigin noticed that the lattice "bosonization" rule for Virasoro algebra can be obtained from the solution of some kind of difference equations in one unknown f with non-commutative coefficients composed of functions of n independent variables $x_1, x_2, \cdots, x_n$ which do not contain the unknown function f. At the time of publishing the work by Yaroslav Pugai [43], no one knew any way to solve similar equations for W-algebras associated to other simple Lie algebras, but in [44, 45] we have shown on the examples that how the classical limit consideration can help to find the right solution. For to do this, we defined a new Poisson bracket based on the Cartan matrix A_n of $\mathfrak{sl}_n$. For example in the case of $\mathfrak{sl}_2$ we define our Poisson bracket as follows

$$\begin{cases} \{X_i, X_j\} := 2X_iX_j & \text{if } i < j \\ \{X_i, X_i\} := 0. \end{cases} \tag{1.19}$$

As we mentioned above, the main problem is to find solutions of the system of difference equations from infinite number of non-commutative variables in quantum case and commutative variables in classical case. It is significant that commutation relations (1.19) depend just on the sign of the difference $(i - j)$ and is based on our Cartan matrix. We should try to find all solutions of the system:

$$\begin{cases} \mathfrak{D}_x^{(n)} \triangleleft \tau_1 = 0 \\ H_x^{(n)} \triangleleft \tau_1 = 0. \end{cases} \tag{1.20}$$

Where $\tau_1 := \tau_1[\cdots, X_1^{(11)}, X_1^{(21)}, X_1^{(31)}, \cdots, X_2^{(12)}, X_2^{(22)}, X_2^{(32)}, \cdots]$, a multi-variable function depend on $\{X_i^{(ji)}\}$'s for $i, j \in \{-\infty, \cdots, 1, \cdots, n, \cdots, +\infty\}$ and $\mathfrak{D}_x^{(n)}$ comes from

$$\{S_{X_i^{ji}}, \tau_1\}_p = S_{X_i^{ji}}\tau_1 - p^{\deg \tau_1 <\alpha_i, \alpha_j>}\tau_1 S_{X_i^{ji}} \tag{1.21}$$

for system of variables X_i^{ji} equiped with lexicographic ordering i.e. $j_{k_m}i < j_{k_n}i$ if $j_{k_m} < j_{k_n}$ and $ji_{k_m} < ji_{k_n}$ if $i_{k_m} < i_{k_n}$ and where $< \alpha_i, \alpha_j >= a_{ij}$ which is related to our Cartan matrix and $S_{X_i^{ji}}$ is the screening operator on one of our variable sets, i.e. $S_{X_i^{ji}} = \sum_{j\in\mathbb{Z}} X_i^{ji}$. And $H_x^{(n)}$ means that the degree of the main solution has to be zero.

By solving these system of differential equations, we were able to compute the main dependent nontrivial solution for lattice W_2, W_3, and up to W_n algebras.

For example in the case of lattice W_n algebras, the functional dependent nontrivial solution for the whole system of the first order partial differential equations will be as follows:

$$\tau_1^{(n)} = \frac{(\Sigma_{1\leq i_1\leq i_2\cdots\leq i_{n-1}\leq 2}\ x_{i_1}^{(1)}x_{i_2}^{(2)}\cdots x_{i_{n-1}}^{(n-1)})(\Sigma_{1\leq i_1\leq i_2\cdots\leq i_{n-1}\leq 2}\ x_{i_1+1}^{(1)}x_{i_2+1}^{(2)}\cdots x_{i_{n-1}+1}^{(n-1)})}{x_2^{(1)}\cdots x_2^{(n-1)}(\Sigma_{1\leq i_1\leq i_2\cdots\leq i_{n-1}\leq 3}\ x_{i_1}^{(1)}x_{i_2}^{(2)}\cdots x_{i_{n-1}}^{(n-1)})}. \tag{1.22}$$

We should notice that $x_{i_j}^{(j)}$s are different of each other for any $j \in \{1, \cdots n-1\}$.

Author's works on the topic of Lattice W-algebras:

References

[1] Razavinia, Farrokh. *Local coordinate systems on quantum flag manifolds.* Чебышевский сборник 21, no. 4 (2020): 171–195; ISSN: 2226-8383.

[2] Razavinia, Farrokh. *Weak Faddeev-Takhtajan-Volkov algebras; Lattice W_n algebras.* Чебышевский сборник 22, no. 1 (2020): 273–291; ISSN: 2226-8383.

1.3. **Bergman's centralizer theorem.** Let X be a set of noncommuting variables, which may or may not be finite, and $\mathbb{F}$ be a field. Let X^* denote the free monoid generated by X. An element of X (resp. X^*) is also called a letter (resp. word) and X is called an alphabet. Let $\mathbb{F}\langle\langle X\rangle\rangle$ and $\mathbb{F}\langle X\rangle$ denote the $\mathbb{F}$-algebra of formal series and polynomials in X, respectively. So an element of $\mathbb{F}\langle\langle X\rangle\rangle$ is in the form $a = \sum_{\omega\in X^*} a_\omega \omega$, where $a_\omega \in \mathbb{F}$ is the coefficient of the word ω in a. The length $\mid \omega \mid$ of $\omega \in X^*$ is the number of letters appearing in ω. For example, if $X = \{x_i\}$ and $\omega = x_1 x_2^2 x_1 x_3$, then $\mid \omega \mid= 5$. Now, we define the valuation

$$\nu : \mathbb{F}\langle\langle X\rangle\rangle \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

as follows: $\nu = \infty$ and if $a = \sum_{\omega\in X^*} a_\omega \omega \neq 0$, then $\nu(a) = \min\{\mid \omega \mid : a_\omega \neq 0\}$. Note that if w is constant, then $\nu(\omega) = 0$ and $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in \mathbb{F}\langle\langle X\rangle\rangle$. The following fact is easy to prove.

Lemma 1.23 (Levi's Lemma). *Let $\omega_1, \omega_2, \omega_3, \omega_4 \in X^*$ be nonzero with $|\omega_2| \geq |\omega_4|$. If $\omega_1\omega_2 = \omega_3\omega_4$, then $\omega_2 = \omega\omega_4$ for some $\omega \in X^*$.*

Next lemma extends Levi's lemma to $k\langle\langle X\rangle\rangle$.

Lemma 1.24 ([34], Lemma 9.1.2). *Let $a, b, c, d \in \mathbb{F}\langle\langle X\rangle\rangle$ be nonzero. If $\nu(a) \geq \nu(c)$ and $ab = cd$, then $a = cq$ for some $q \in \mathbb{F}\langle\langle X\rangle\rangle$.*

An interesting consequence of Lemma 1.24 is the following result:

Corollary 1.25. *Let $a \in \langle\langle X\rangle\rangle$. Then $b \in C(a; \mathbb{F}\langle\langle X\rangle\rangle)$ if and only if a, b are not free, i.e. $f(a, b) = 0$ for some nonzero series $f \in \mathbb{F}\langle\langle x, y\rangle\rangle$.*

Lemma 1.26. *Suppose that the constant term of an element $a \in \mathbb{F}\langle\langle X\rangle\rangle$ is zero and $b, c \in C(a; \mathbb{F}\langle\langle X\rangle\rangle)\ \{0\}$. If $\nu(c) \geq \nu(b)$, then $c = bd$ for some $d \in C(a; \mathbb{F}\langle\langle X\rangle\rangle)$.*

With the help of the preceding lemmas, we can state this well-known centralizer theorem of $\mathbb{F}$-algebra of formal series by Cohn.

Theorem 1.27 (Cohn's Centralizer Theorem, [17]). *If $a \in \mathbb{F}\langle\langle X\rangle\rangle$ is not a constant, then the centralizer $C(a; k\langle\langle X\rangle\rangle) \cong \mathbb{F}[\![x]\!]$, where $\mathbb{F}[\![x]\!]$ is the algebra of formal power series in the variable x.*

Now since $\mathbb{F}\langle X\rangle \subset \mathbb{F}\langle\langle X\rangle\rangle$, it follows from the above theorem that if $a \in \mathbb{F}\langle X\rangle$ is not constant, then $C(a; \mathbb{F}\langle X\rangle)$ is commutative because $C(a; \mathbb{F}\langle\langle X\rangle\rangle)$ is commutative. The next theorem is our main goal which shows that there is a similar result for $C(a; \mathbb{F}\langle X\rangle)$.

Theorem 1.28 (Bergman's Centralizer Theorem, [6]). *If $a \in \mathbb{F}\langle X\rangle$ is not constant, then the centralizer $C(a; \mathbb{F}\langle X\rangle) \cong \mathbb{F}[x]$, where $\mathbb{F}[x]$ is the polynomial algebra in one variable x.*

Here in this thesis, we will not fully restate the original proof of Bergman's centralizer theorem since this is not the main idea here. However, we do use a result in his original proof [6] which helps us to finish the proof of the fact that the centralizer is integrally closed.

By the opinion of most specialists, including E. Rips, there are no new proofs of Bergman's centralizer theorem [6] for almost fifty years. We use a method of deformation quantization presented by Kontsevich to give an alternative proof of Bergman's centralizer theorem. First we get that the centralizer is a commutative domain of transcendence degree one.

Theorem 1.29. *Let A, B be two commuting generic matrices in $\mathbb{F}\langle X_1, \ldots, X_s\rangle$ with $\mathsf{rank}\mathbb{F}\langle A, B\rangle = 2$, and let $\hat{A}$ and $\hat{B}$ be quantized images (by sending multiplications to star products by means of Kontsevich's formal quantization) of A and B respectively by considering lifting A and B in $\mathbb{F}\langle X_1, \ldots, X_s\rangle[\![\mathfrak{h}]\!]$. Then $\hat{A}$ and $\hat{B}$ do not commute. Moreover,*

$$\frac{1}{\mathfrak{h}}[\hat{A}, \hat{B}]_\star \equiv \begin{pmatrix} \frac{1}{\mathfrak{h}}\{\lambda_1, \mu_1\} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\mathfrak{h}}\{\lambda_n, \mu_n\} \end{pmatrix} \mod \mathfrak{h}, \tag{1.30}$$

where λ_i and μ_i are eigenvalues(weights) of A and B, respectively.

We need the following Lemma:

Lemma 1.31. *Let $\hat{A} \equiv A_0 + \mathfrak{h}A_1 (\operatorname{mod} \mathfrak{h}^2)$ be the quantized image of a generic matrix $A \in \mathbb{F}\langle X_1, \ldots, X_s\rangle$, where A_0 is diagonal with distinct eigenvalues. Then, the quantized images $\hat{A}$ can be diagonalized over some finite extension of $\mathbb{F}[x_{ij}^{(\nu)}]$.*

Now A, B are two algebraically independent but commuting generic matrices in $\mathbb{F}\langle X_1, \ldots, X_s\rangle$. Hence we may assume A and B can be both diagonalized over an integral extension of $\mathbb{F}[x_{ij}^{(\nu)}]$. Consider the result of diagonalization in $\mathbb{F}\langle X_1, \ldots, X_s\rangle[\![\mathfrak{h}]\!]$ and then we compute the quantization commutator of two quantized generic matrices over $\mathbb{F}\langle X_1, \ldots, X_s\rangle[\![\mathfrak{h}]\!]$.

This leads to a contradiction to Theorem 1.29 which shows that $[\hat{A}, \hat{B}]_\star \neq 0$. So we obtain the following result.

Theorem 1.32. *There are no commutative subalgebras of rank ≥ 2 in the free associative algebra $\mathbb{F}\langle X\rangle$.* □

Hence, so far by the above results we just was able to show that the centralizer C is a commutative domain of transcendence degree one. However, we have to prove the fact that C is integrally closed in order to complete the proof of Bergman's Centralizer Theorem.

By Cohn's centralizer theorem, the centralizer of every nonconstant element in $\mathbb{F}\langle\langle X\rangle\rangle$ is commutative and since $\mathbb{F}\langle X\rangle$ is a $\mathbb{F}$-subalgebra of $\mathbb{F}\langle\langle X\rangle\rangle$, the centralizer of a nonconstant element of $\mathbb{F}\langle X\rangle$ will be commutative as well.

Bergman proved that if $f \in \mathbb{F}\langle X\rangle$ is not constant, then $C(f; \mathbb{F}\langle X\rangle)$ is integrally closed. He used this result to prove that $C(f; \mathbb{F}\langle X\rangle) = \mathbb{F}[\ell]$ for some $h \in \mathbb{F}\langle X\rangle$. This is called *Bergman's centralizer theorem.* Next, we will give a proof of the following main theorem by using the generic matrices technique:

Theorem 1.33. *The centralizer C of a non-trivial element f in the free associative algebra is integrally closed.*

we have the following proposition:

Proposition 1.34. *Let p be a large enough prime number, and $\mathbb{F}\{X\}$ the algebra of generic matrices of order p. For any $A \in \mathbb{F}\{X\}$, the centralizer of A is rationally closed and integrally closed in $\mathbb{F}\{X\}$ over the center of $\mathbb{F}\{X\}$.*

Now we need one fact from Bergman [6]. Let X be a totally ordered set, W be the free semigroup with identity 1 on set X. We have the following lemma.

Lemma 1.35 (Bergman)**.** *Let $u, v \in W \setminus \{1\}$. If $u^\infty > v^\infty$, then we have $u^\infty > (uv)^\infty > (vu)^\infty > v^\infty$.*

Proposition 1.36 (Bergman)**.** *If $C \neq \mathbb{F}$ is a finitely generated subalgebra of $\mathbb{F}\langle X\rangle$, then there is a homomorphism f of C into the polynomial algebra over $\mathbb{F}$ in one variable, such that $f(C) \neq \mathbb{F}$.*

And by using these results we can finish the proof of Bergman's centralizer theorem.

Author's works on the topic of Bergman's centralizer theorem:

References

[1] Kanel Belov, Alexei, Farrokh Razavinia, and Wenchao Zhang. *Bergman's Centralizer Theorem and quantization.* Communications in Algebra 46.5 (2017): 2123-2129.

[2] Kanel-Belov, Alexei, Farrokh Razavinia, and Wenchao Zhang. *Centralizers in free associative algebras and generic matrices.* arXiv preprint arXiv:1812.03307 (2018).

1.4. **Algebraic torus actions.** Let us first restrict ourselves to the situation in which an r-dimensional torus $\mathbb{T}_r = (\mathbb{G}_m)^r \simeq (\mathbb{F}^*)^r$ acts on an affine n-space $\mathbb{A}^n := Spec\mathbb{F}[x_1, \cdots, x_n]$, where $\mathbb{F}[x_1, \cdots, x_n] = \mathbb{F}^{[n]}$ is an n-variable polynomial ring over $\mathbb{F}$. Since the quotient of a torus by any subgroup is again a torus we may assume that $\mathbb{T}_r$ acts effectively, i.e. that no proper subgroup of $\mathbb{T}_r$ acts neutrally on $\mathbb{A}^n$.

In [9, 10], Białynicki-Birula proved the following results, for $\mathbb{F}$ algebraically closed.

Theorem 1.37. *Any regular action of $\mathbb{T}_n$ on $\mathbb{A}^n$ has a fixed point.*

Theorem 1.38. *Any effective and regular action of $\mathbb{T}_n$ on $\mathbb{A}^n$ is a representation in some coordinate system.*

Theorem 1.39. *The action of $\mathbb{T}_r$ on $\mathbb{A}^n$ is linearizable in the cases of $r = n$ or $r = n-1$, which means that one can find isobaric elements $y_1, \cdots, y_n$ in $\mathbb{F}^{[n]} = \mathbb{F}[x_1, \cdots, x_n]$ such that $\mathbb{F}^{[n]} = \mathbb{F}[y_1, \cdots, y_n]$.*

We then establish the free algebra version of the Białynicki-Birula theorem. The latter is formulated as follows.

Theorem 1.40. *Suppose given an action σ of the algebraic n-torus $\mathbb{T}_n$ on the free algebra F_n. If σ is effective, then it is linearizable.*

The linearity (or linearization) problem, as it has become known since Kambayashi, asks whether all (effective, regular) actions of a given type of algebraic groups on the affine space of given dimension are conjugate to representations. According to Theorem 1.40, the linearization problem extends to the noncommutative category. Several known results concerning the (commutative) linearization problem are summarized below.

(1) Any effective regular torus action on $\mathbb{A}^2$ is linearizable (Gutwirth [29]).
(2) Any effective regular torus action on $\mathbb{A}^n$ has a fixed point (Białynicki-Birula [9]).
(3) Any effective regular action of $\mathbb{T}_{n-1}$ on $\mathbb{A}^n$ is linearizable (Białynicki-Birula [10]).
(4) Any (effective, regular) one-dimensional torus action (i.e., action of $\mathbb{F}^*$) on $\mathbb{A}^3$ is linearizable (Koras and Russell [33]).
(5) If the ground field is not algebraically closed, then a torus action on $\mathbb{A}^n$ need not be linearizable. In [1], Asanuma proved that over any field $\mathbb{F}$, if there exists a non-rectifiable closed embedding from $\mathbb{A}^m$ into $\mathbb{A}^n$, then there exist non-linearizable effective actions of $(\mathbb{F}^*)^r$ on $\mathbb{A}^{1+n+m}$ for $1 \le r \le 1+m$.
(6) When $\mathbb{K}$ is infinite and has positive characteristic, there are examples of non-linearizable torus actions on $\mathbb{A}^n$ (Asanuma [1]).

we can give the following linearization problem, which we formulate as a conjecture.

Conjecture 1.41. *For $n \ge 1$, let P_n denote the commutative Poisson algebra, i.e., the polynomial algebra*

$$\mathbb{F}[z_1, \ldots, z_{2n}]$$

equipped with the Poisson bracket defined by

$$\{z_i, z_j\} = \delta_{i,n+j} - \delta_{i+n,j}.$$

Then any effective regular action of $\mathbb{T}_n$ by automorphisms of P_n is linearizable.

It is nonetheless possible that the free analogue of the main result of [10] exists. We have then the following conjecture.

Conjecture 1.42. *Any effective regular action of $\mathbb{T}_{n-1}$ on the free algebra $F_n(\mathbb{F})$ is linearizable, provided that $\mathbb{F}$ is algebraically closed.*

And according to above results we are now able to state and prove one of our main results:

Theorem 1.43. *Let $\mathbb{F}$ be algebraically closed. Any effective regular action of (the one-dimensional torus) $\mathbb{F}^*$ on the free algebra $\mathbb{F}\langle z_1, z_2\rangle$ is linearizable.*

Next we consider positive-root torus actions and prove the linearity property analogous to the Białynicki-Birula theorem.

Theorem 1.44. *Any effective positive-root action of $\mathbb{T}_r$ on $\mathbb{F}[x_1, \ldots, x_n]$ is linearizable.*

In order to prove the free associative version of this theorem, we devise a way to reduce the positive-root case to the commutative one. To that end, we introduce the generic matrices and induce the action on the rings of coefficients.

More precisely, we have the following.

Theorem 1.45. *Let $\sigma : \mathbb{T}_r \times F_n \to F_n$ be a regular torus action with positive roots. Then it is linearizable.*

Next we study non-linearizable torus actions. The examples of non-linearizable torus actions, as well as a way to study them, were developed by Asanuma [1]. It is not difficult to observe that most of Asanuma's technique can be carried to the free associative case without loss of generality. As in Asanuma's case, the existence of non-linearizable torus actions is tied to the existence of so-called non-rectifiable ideals in the appropriate algebras. One rather remarkable feature of Asanuma's technique is the fact that, modulo minor details and replacements, it may be repeated almost verbatim in the associative category – a situation similar to the one we have observed in the Białynicki-Birula theorem on the action of the maximal torus.

Definition 1.46. Two (regular) $\mathbb{T}_r$-actions ϕ and ψ, respectively, on A and B are equivalent, if there exists a $\mathbb{F}$-homomorphism $\sigma : A \to B$, such that the diagram

$$\begin{tikzcd} A[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}] \arrow[r, "\sigma\otimes Id"] & B[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}] \\ A \arrow[u, "\phi"] \arrow[r, "\sigma"] & B \arrow[u, "\psi"] \end{tikzcd}$$

commutes.

Proposition 1.47. *A regular $\mathbb{T}_r$-action ϕ on A is linearizable (in the sense of the previous sections), if and only if it is equivalent, in the sense of Definition 1.46, to a linear action on A. (An action $\psi : A \to A[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}]$ is linear if the images $\psi(x_i)$ of the generators of A are linear in x_i.)*

The main problem of interest is the free associative analogue of the so-called Cancellation Conjecture, as formulated by V. Drensky and Yu [18]:

Conjecture 1.48. *Let R be a $\mathbb{F}$-algebra. If*

$$R * \mathbb{F}\langle y\rangle \simeq_{\mathbb{F}} \mathbb{F}\langle x_1, \ldots, x_n\rangle,$$

then

$$R \simeq_{\mathbb{F}} \mathbb{F}\langle x_1, \ldots, x_{n-1}\rangle.$$

Asanuma's results on the Rees algebras allow us to establish a version of the Cancellation Conjecture for co-products over a (commutative) $\mathbb{F}$-algebra D. The following statement holds.

Theorem 1.49. *Let D be an integral domain which is a $\mathbb{F}$-algebra, and let x be an indeterminate over D. Given a non-zero element $t \in D$ and monic polynomials $f(x)$ and $g(x)$ in $\mathbb{F}[x]$ of degree greater than 1. Set $A = D[x, t^{-1}f(x)]$ and $B = D[x, t^{-1}g(x)]$. If*

$$\mathbb{F}[x]/(f(x)) \simeq_{\mathbb{F}} \mathbb{F}[x]/(g(x)),$$

then

$$A *_D \mathbb{F}\langle y\rangle \simeq_D B *_D \mathbb{F}\langle y\rangle,$$

where the product $R *_D S$ is the quotient of the free product $R * S$ over $\mathbb{F}$ by the ideal generated by all elements of the form

$$r * (ds) - d(r * s).$$

And we have the following conjectures:

One notable example is that we expect the free associative analogue of the second Białynicki-Birula theorem to hold and formulate it here as a conjecture.

Conjecture 1.50. *Any effective action of $\mathbb{T}_{n-1}$ on F_n is linearizable.*

Also of independent interest is the following instance of the linearity problem.

Conjecture 1.51. *For $n \geq 1$, let P_n denote the commutative Poisson algebra, i.e., the polynomial algebra $\mathbb{F}[z_1, \ldots, z_{2n}]$ equipped with the Poisson bracket defined by*

$$\{z_i, z_j\} = \delta_{i,n+j} - \delta_{i+n,j}.$$

Then any effective regular action of $\mathbb{T}_n$ by automorphisms of P_n is linearizable.

This problem is loosely analogous to the Białynicki-Birula theorem, in the sense of maximality of torus with respect to the dimension of the configurations space (spanned by x_i).

Author's works on the topic of algebraic torus actions:

References

[1] Belov-Kanel, Alexei Yakovlevich, Andrey Mikhailovich Elishev, Farrokh Razavinia, Jie-Tai Yu, and Zhang Wenchao. *Noncommutative Bialynicki-Birula Theorem.* Чебышевский сборник 21, no. 1 (2020): 51-61; ISSN: 2226-8383.

[2] Elishev, Andrey, Alexei Kanel-Belov, Farrokh Razavinia, Jie-Tai Yu, and Wenchao Zhang. *Torus actions on free associative algebras, lifting and Biał ynicki-Birula type theorems.* arXiv preprint arXiv:1901.01385 (2019).

The scientific novelty of the work:

All results of the dissertation are prepared personally by the applicant under the scientific supervision of the Doctor of Physical and Mathematical Sciences, Professor Alexei Kanel-Belov and all are new.

- We propose a very efficient way of solving a system of q-linear homogeneous differential equations in one unknown f with coefficients composed of functions of n independent variables $x_1, x_2, ..., x_n$ which do not contain the unknown function f.
- We propose a very efficient way of solving a system of q-linear homogeneous differential equations in one unknown f with coefficients composed of functions of n independent variables $x_1, x_2, ..., x_n$ which do not contain the unknown function f.
- We present a new concept of looking at quantized algebras by defining our new class of algebras by showing their connectivity to the already known class of algebras.
- We propose a new conceptual proof of an old and known theorem, Bergman's centralizer theorem, which hasn't been revisited for almost 50 years.
- We study a free algebra analogue of a classical theorem of Białynicki-Birula's theorem and give a noncommutative version of this famous theorem. We also consider positive-root torus actions and prove the linearity property analogous to the Białynicki-Birula theorem.

Presentations and validation of research results.

The main results of the work were reported at the following scientific conferences and seminars:

1 conference on modern methods, problems and applications of operator theory and harmonic analysis VII, Talk on "Weak Faddeev-Takhtajan-Volkov algebras" Rostov-on-Don, Russia - Southern Federal University, 2017.

2 Ph.D. student seminar; October 15th, 2017, "Weak Faddeev-Takhtajan-Volkov alge bras"; Porto, Portugal, Department of mathematics, University of Porto.

3 Summer School of the UC|UP Joint Ph.D. Program in Mathematics, University of Aveiro, mathematics department; September 3-14th, 2018, "Generalized Heisenberg Algebras and their Poisson semiclassical limit".

4 Moscow-Beijing Topology Seminar, Online via Zoom; February 17th, 2021, "On the algebraic structures of the quantum generalized Heisenberg algebras".

5 The 86th seminaire Lotharingien de Combinatoire, Evangelische Akademie, Bad Boll, Germany; 2021, September 5-8th, "Quantum generalized Heisenberg algebras and their combinatorial trace".

The content of the work:

The thesis consists of the Conventions and the organization of the thesis, introduction, and five more chapters, which can be summarized as follows:

Chapter 2 is devoted to the preliminaly results and definitions, which starts from groups, rings, algebras and modules to Ore extensions, ambiskew polynomial rings, generalized Weyl algebras, (generalized) down-up algebras, generalized Heisenberg algebras, semisimple Lie algebras, Kac-Moody Lie algebras, Hopf algebras, quantum groups, deformation quantization, Jacobian, Dixmier and Kontsevich conjectures, approximation, torus actions and Biały nicki-Birula theorem.

In chapter 3 we will introduce Feigin's homomorphisms and we will explore their relations with screening operators. Then we turn to local integral of motions; Volkov's scheme and lattice Virasoro algebras. In Section **??** we will introduce lattice W_2 algebra, W_3 algebra, up to W_n algebras based on our newly defined Poisson bracket which has been obtained by using the Cartan matrix A_n.

Chapter 4 is devoted to the history of the creation of generalized Heisenberg algebras and the advent of quantum generalized Heisenberg algebras.

Chapter 5 starts off with the definition of the quantum generalized Heisenberg algebras $\mathcal{H}_q(f,g)$ and then in Section 5.2 we relate quantum generalized Heisenberg algebras to known constructions, such as Ore extensions, ambiskew polynomial rings and generalized Weyl algebras. From these we deduce the basic properties of the algebras $\mathcal{H}_q(f,g)$, including a PBW-type basis, necessary and sufficient conditions for $\mathcal{H}_q(f,g)$ to be a domain. By using an appropriate filtration and results on Gelfand-Kirillov dimension, we are able to prove in Corollary 5.75 that if $\mathsf{deg}\, f > 1$ then $\mathcal{H}_q(f,g)$ is not isomorphic to a generalized down-up algebra. This divides the class of $qGHA$ into two natural subclasses: if $\mathsf{deg}\, f \leq 1$ we get all generalized down-up algebras, which have been extensively studied from many points of view; if $\mathsf{deg}\, f > 1$ we get algebras which are non-Noetherian domains (as long as $q \neq 0$) and which, in spite of appearing to be of a similar nature, have not been yet studied in depth, as far as we know.

In Section 5.2 we also characterize the Noetherian quantum generalized Heisenberg algebras. While it is well known that for generalized down-up algebras being Noetherian is equivalent to being a domain ([32], [13]), we see that within our wider class of algebras this correspondence no longer holds as for $q \neq 0$ and $\mathsf{deg}\, f > 1$ the algebra $\mathcal{H}_q(f,g)$ will be a non-Noetherian domain.

The isomorphism problem for quantum generalized Heisenberg algebras is tackled in Section 5.3 and it will be seen that the isomorphism relation can be phrased in very concrete geometric terms, very much like in [4]. It will follow in particular that, in case $q \neq 0$ and $\mathsf{deg}\, f > 1$, the parameter q, as well as the integers $\mathsf{deg}\, f$ and $\mathsf{deg}\, g$, are invariant under isomorphism, showing that $qGHA$ are indeed a vast generalization of generalized Heisenberg algebras and generalized down-up algebras.

In Section 5.4, we classify, up to isomorphism, all finite-dimensional simple representations of $\mathcal{H}_q(f,g)$, assuming only that $q \neq 0$ and that the base field is algebraically closed, although of arbitrary characteristic. We also explicitly describe all possible isomorphisms between these modules.

In terms of automorphism groups, which we study in Section 5.6, an interesting phenomenon occurs. Although, as long as either $\mathsf{char}(\mathbb{F}) = 0$ or $\mathsf{char}(\mathbb{F}) \neq \mathsf{deg}\, f$, the automorphism group of a quantum generalized Heisenberg algebra $\mathcal{H}_q(f,g)$ with $q \neq 0$ and $\mathsf{deg}\, f > 1$ is Abelian and does not depend on the parameter q (although its isomorphism class does), if we allow $\mathsf{char}(\mathbb{F}) = \mathsf{deg}\, f$ then we can obtain non-Abelian automorphism groups.

In section 5.7, we will investigate the Gelfand–Kirillov dimension of quantum generalized Heisenberg algebras $\mathcal{H}_q(f,g)$.

In section 5.8, if we let $\mathbb{F}$ be a field of prime characteristic p; we will try to define a Hopf algebra structure on the quantum generalized Heisenberg algebras $\mathcal{H}_q(f,g)$ in the case where $f(h) = h + b$ for any $b \in \mathbb{F}$ or $f(h) = qh$ and $g(h) \in \mathsf{span}_{\mathbb{F}}\{h^{p^k} : k \geq 0\}$. After that we will study the tensor product of their simple modules.

In Chapter 6 we first give a brief summary of the background of two well-known centralizer theorems in the power series ring and in the free associative algebra, i.e. Cohn's centralizer theorem and Bergman's centralizer theorem and then in Section 6.2 we will establish the relation between the algebra of generic matrices and the commutative subalgebras in the free associative algebra. In Section 6.3 we give an alternative proof of Bergman's centralizer theorem by using a method of deformation quantization presented by Kontsevich and will complete this proof in Section 6.4 by proving the fact that the centralizer C is integrally closed.

Chapter 7 starts by giving some introductury facts on torus actions and Białynicki-Birula type theorems and then in Section 7.2 we give a proof of the free algebra analogue of a classical theorem of Białynicki-Birula (Theorem 7.5), which intuitively states that every maximal torus action on the free algebra is conjugate to a linear action. In the Section 7.4, we consider positive-root torus actions and prove the linearity property analogous to the Białynicki-Birula theorem and finally in Section 7.5 we study the non-linearizable torus actions.

Author's publications on the dissertation topic

Main papers:

1 **Razavinia, Farrokh**. *Local coordinate systems on quantum flag manifolds.* Чебышевский сборник 21, no. 4 (2020): 171–195; ISSN: 2226-8383.

2 **Razavinia, Farrokh**. *Weak Faddeev-Takhtajan-Volkov algebras; Lattice W_n algebras.* Чебышевский сборник 22, no. 1 (2020): 273–291; ISSN: 2226-8383.

3 Kanel Belov, Alexei, **Farrokh Razavinia**, and Wenchao Zhang. *Bergman's Central izer Theorem and quantization.* Communications in Algebra 46.5 (2017): 2123-2129.

4 Belov-Kanel, Alexei Yakovlevich, Andrey Mikhailovich Elishev, **Farrokh Razavinia**, Jie-Tai Yu, and Zhang Wenchao. *Noncommutative Bialynicki-Birula Theorem.* Чебышевский сборник 21, no. 1 (2020): 51-61; ISSN: 2226-8383.

5 Lopes, Samuel A., and **Farrokh Razavinia**. *Quantum generalized Heisenberg algebras and their representations.* Communications in Algebra (2021): 1-21.

6 Lopes, Samuel A., and **Farrokh Razavinia**. *Structure and isomorphisms of quantum generalized Heisenberg algebras.* Journal of Algebra and its Applications (2021).

Other publications:

7 Kanel-Belov, Alexei, **Farrokh Razavinia**, and Wenchao Zhang. *Centralizers in free associative algebras and generic matrices.* arXiv preprint arXiv:1812.03307 (2018).

8 Kanel-Belov, Alexei, Andrey Elishev, **Farrokh Razavinia**, Jie-Tai Yu, and Wenchao Zhang. *Polynomial automorphisms, quantization and Jacobian conjecture related prob lems.* arXiv preprint arXiv:1912.03759 (2019).

9 Elishev, Andrey, Alexei Kanel-Belov, **Farrokh Razavinia**, Jie-Tai Yu, and Wenchao Zhang. *Torus actions on free associative algebras, lifting and Biał ynicki-Birula type theorems.* arXiv preprint arXiv:1901.01385 (2019).

References

[1] Asanuma, Teruo. *Non-linearizable algebraic k^*-actions on affine spaces.* Inventiones mathematicae 138.2 (1999): 281.

[2] V. Bavula and F. van Oystaeyen. *The simple modules of certain generalized crossed products.* J. Algebra, 194(2):521–566, 1997.

[3] V. V. Bavula. *Finite-dimensionality of* Ext^n *and* Tor_n *of simple modules over a class of algebras.* Funktsional. Anal. i Prilozhen., 25(3):80–82, 1991.

[4] V. V. Bavula and D. A. Jordan. *Isomorphism problems and groups of automorphisms for generalized Weyl algebras.* Trans. Amer. Math. Soc., 353(2):769–794, 2001.

[5] G. Benkart and T. Roby. *Down-up algebras.* J. Algebra, 209(1):305–344, 1998.

[6] Bergman, G. M. (1969). *Centralizers in free associative algebras.* Transactions of the American Mathematical Society, 137, 327-344.

[7] Bershadsky, Michael, and Hirosi Ooguri. *Heidden SL (n) symmetry in conformal field theories.* Communications in Mathematical Physics 126.1 (1989): 49-83.

[8] Belavin, A. A. *KdV-type equations and W-algebras.* Integrable systems in quantum field theory and statistical mechanics, 117–125, Adv. Stud. Pure Math., 19, Academic Press, Boston, MA, 1989.

[9] Białynicki-Birula, A. *Remarks on the action of an algebraic torus on k.* Bull. Pol. Acad. Sci., Math 14 (1966): 177-181.

[10] Białynicki-Birula, A. *Remarks on the actionof an algebraic torus on kn, II.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys 15 (1967): 123-125.

[11] Bouwknegt, Peter, Jim McCarthy, and Krzysztof Pilch. *Some aspects of free field resolutions in 2D CFT with application to the quantum Drinfeld-Sokolov reduction.* arXiv preprint hep-th/9110007 (1991).

[12] Richard E. Block. *The irreducible representations of the Lie algebra* $\mathfrak{sl}(2)$ *and of the Weyl algebra.* Adv. in Math., 39(1):69–110, 1981.

[13] Thomas Cassidy and Brad Shelton. *Basic properties of generalized down-up algebras.* J. Algebra, 279(1):402–421, 2004.

[14] Eun-Hee Cho and Sei-Qwon Oh. *Semiclassical limits of Ore extensions and a Poisson generalized Weyl algebra.* Lett. Math. Phys., 106(7):997–1009, 2016.

[15] Sergio Chouhy, Estanislao Herscovich, and Andrea Solotar. *Hochschild homology and cohomology of down-up algebras.* J. Algebra, 498:102–128, 2018.

[16] E M F Curado and M A Rego-Monteiro. *Multi-parametric deformed heisenberg algebras: a route to complexity.* Journal of Physics A: Mathematical and General, 34(15):3253, 2001.

[17] Cohn, P. M. *Subalgebras of free associative algebras.* Proceedings of the London Mathematical Society 3, no. 4 (1964): 618-632.

[18] Drensky, Vesselin, and Jie-Tai Yu. *A cancellation conjecture for free associative algebras.* Proceedings of the American Mathematical Society 136.10 (2008): 3391-3394.

[19] Dotsenko, Vl. S. *Lectures on conformal field theory.* Conformal field theory and solvable lattice models (Kyoto, 1986), 123–170, Adv. Stud. Pure Math., 16, Academic Press, Boston, MA, 1988.

[20] Drinfel'd, Vladimir G., and Vladimir V. Sokolov. *Lie algebras and equations of Korteweg-de Vries type.* Journal of Soviet mathematics 30.2 (1985): 1975-2036.

[21] Faddeev, L. D.; Takhtajan, L. A. *Liouville model on the lattice.* Field theory, quantum gravity and strings (Meudon/Paris, 1984/1985), 166–179, Lecture Notes in Phys., 246, Springer, Berlin, 1986.

[22] Fateev, VA, and AB Zamolodchikov. *Conformal quantum field theory models in two dimensions having Z3 symmetry.* Nuclear Physics B 280 (1987): 644-660.

[23] Fateev, Vladimir A., and S. L. Lykyanov. *The models of two-dimensional conformal quantum field theory with Zn symmetry.* International Journal of Modern Physics A 3.02 (1988): 507-520.

[24] Fateev, VA, and SL Lukyanov. *Poisson–Lie groups and classical W-algebras.* International Journal of Modern Physics A 7.05 (1992): 853-876.

[25] Fateev, VA, and SL Lukyanov. *Vertex operators and representations of quantum universal enveloping algebras.* International Journal of Modern Physics A 7.07 (1992): 1325-1359.

[26] Feigin, Boris, and Edward Frenkel. *Quantization of the Drinfeld-Sokolov reduction.* Physics Letters B 246.1-2 (1990): 75-81.

[27] Siân Fryer. *The prime spectrum of quantum SL_3 and the Poisson prime spectrum of its semiclassical limit.* Trans. London Math. Soc., 4(1):1–29, 2017.

[28] K. R. Goodearl. *Semiclassical limits of quantized coordinate rings.* Advances in ring theory, Trends Math., pages 165–204. Birkhäuser/Springer Basel AG, Basel, 2010.

[29] Gutwirth, A. *The action of an algebraic torus on the affine plane.* Transactions of the American Mathematical Society 105.3 (1962): 407-414.

[30] Jeff Hildebrand. *Irreducible finite-dimensional modules of prime characteristic down-up algebras.* J. Algebra, 273(1):295–319, 2004.

[31] E.E. Kirkman and L.W. Small. *q-analogs of harmonic oscillators and related rings.* Israel J. Math., 81(1-2):111–127, 1993.

[32] Kirkman, Ellen, Ian Musson, and D. Passman. *Noetherian down-up algebras.* Proceedings of the American Mathematical Society 127.11 (1999): 3161-3167.

[33] Koras, Mariusz, and Peter Russell. *C^*-actions on C^3: The smooth locus of the quotient is not of hyperbolic type.* Journal of Algebraic Geometry 8.4 (1999): 603-694.

[34] Lothaire, M. (1997). Combinatorics on words (Vol. 17). Cambridge university press.

[35] Samuel A. Lopes. *Non-Noetherian generalized Heisenberg algebras.* J. Algebra Appl., 16(2):1750064, 2017.

[36] Lopes, Samuel A., and Farrokh Razavinia. *Quantum generalized Heisenberg algebras and their representations.* Communications in Algebra (2021): 1-21.

[37] Lopes, Samuel A., and Farrokh Razavinia. *Structure and isomorphisms of quantum generalized Heisenberg algebras.* Journal of Algebra and its Applications (2021).

[38] Rencai Lü, Volodymyr Mazorchuk, and Kaiming Zhao. *Simple weight modules over weak generalized Weyl algebras.* J. Pure Appl. Algebra, 219(8):3427–3444, 2015.

[39] Rencai Lü and Kaiming Zhao. *Finite-dimensional simple modules over generalized Heisenberg algebras.* Linear Algebra Appl., 475:276–291, 2015.

[40] Jonathan Nilsson. *Simple $\mathfrak{sl}_{n+1}$-module structures on $\mathfrak{U}(\mathfrak{h})$.* J. Algebra, 424:294–329, 2015.

[41] Matthew Towers. *Poisson and Hochschild cohomology and the semiclassical limit.* J. Noncommut. Geom., 9(3):665–696, 2015.

[42] Mnatsakanova, M. N., and Yu S. Vernov. *Properties of the algebra of q-deformed commutators in indefinite-metric space.* Theoretical and mathematical physics 113.3 (1997): 1497-1507.

[43] Pugay, Ya P. *Lattice W algebras and quantum groups.* Theoretical and Mathematical Physics 100.1 (1994): 900-911.

[44] Razavinia, Farrokh. *Local coordinate systems on quantum flag manifolds.* Чебышевский сборник 21, no. 4 (2020): 171–195; ISSN: 2226-8383.

[45] Razavinia, Farrokh. *Weak Faddeev-Takhtajan-Volkov algebras; Lattice W_n algebras.* Чебышевский сборник 22, no. 1 (2020): 273–291; ISSN: 2226-8383.

[46] Vernov, Yu S., and Melita Nikolaevna Mnatsakanova. *Regular representations of theR-deformed Heisenberg algebra.* Theoretical and Mathematical Physics 125.2 (2000): 1531-1538.

[47] Zamolodchikov, Aleksandr Borisovich. *Infinite additional symmetries in two-dimensional conformal quantum field theory.* Teoreticheskaya i Matematicheskaya Fizika 65.3 (1985): 347-359.